The Topological Triathlon

(minus the topology)

Integral Bee

Problem 1 (4 points)**.** Compute the following integral.

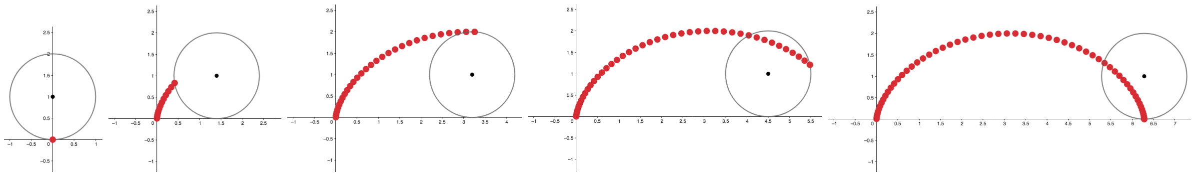
$$\int_{1}^{5} e^{\frac{x}{\lfloor x \rfloor}}\, dx$$

Problem 2 (8 points)**.** Compute the error of a famous approximation of π,

$$\int_{0}^{1} \frac{x^4(1-x)^4}{1+x^2}\, dx.$$

Is this approximation an overestimate or underestimate of the true value of π?

Problem 3 (12 points)**.** Find the area under one arc of the cycloid, a curve traced by a point on a rolling unit circle.



Counting Conundrum

Problem 4 (4 points)**.** How many seven card hands dealt from a standard deck of 52 cards contain at most five clubs?

Problem 5 (8 points)**.** Suppose Flora has 12 pairs of shoes in her closet, each of a different style. One morning, Flora reaches into her closet and randomly pulls out 5 shoes. What is the probability of exactly one pair sitting among the 5?

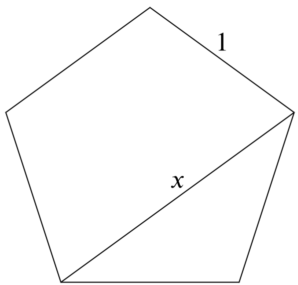
Problem 6 (12 points)**.** Five people are lost in the wilderness with limited resources, so they make alliances with each other to survive. Alliances are symmetric: For example, if John is allied with Jill, then Jill is allied with John. However, alliances are not always transitive: If John is allied with Jill and Jill is allied with Jake, John and Jake might not be allies. What is the smallest possible number of pairs of people on the island with the same number of alliances? (For example, if everyone is allied with everyone else, there are 10 pairs of people, each pair containing two people who both have 4 alliances.)

Mystery Mayhem

Problem 7 (4 points)**.** Ada, Beth, Christine, Julia, and Wanda are identical quintuplets. Ada and Christine always tell the truth, while Beth and Julia only lie. Wanda is known to do both. One day, four of them are eating lunch together and to their amazement, they can't decide who's missing. Their dialog proceeds as follows: "I'm Wanda," says the first. "I'm not Wanda," says the second. "If anybody here is Ada, then I'm Wanda," says the third to the first. The fourth, having recognized one of her siblings, is sure she knows the answer. Quite pleased with herself, she blurts, "nobody's lied yet!" Who's missing?

Problem 8 (8 points)**.** There is a flat, regular-hexagonal storm on the north pole of Saturn whose sides are 31,415 km long. 7 meteorites get sucked into Saturn's gravity and crash into the storm. Find shortest distance X km such that there must be two meteorites no more than X km apart, and prove that it indeed must be the shortest.

Problem 9 (12 points)**.** Find an exact expression (no trig functions!) for the length x in the regular pentagon below.



Solutions

Integral Bee

(1)

$$
\begin{aligned}
\int_1^5 e^{\frac{x}{\lfloor x \rfloor}}\,\mathrm{d}x &= \sum_{k=1}^{4} \int_k^{k+1} e^{\frac{x}{\lfloor x \rfloor}}\,\mathrm{d}x \\
&= \sum_{k=1}^{4} \int_k^{k+1} e^{\frac{x}{k}}\,\mathrm{d}x \\
&= \sum_{k=1}^{4} \left[k e^{\frac{x}{k}} \right]_k^{k+1} \\
&= \sum_{k=1}^{4} k e^{\frac{k+1}{k}} - ke \\
&= -10e + e^2 + 2e^{\frac{3}{2}} + 3e^{\frac{4}{3}} + 4e^{\frac{5}{4}}
\end{aligned}
$$

(2)

$$
\begin{aligned}
\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\mathrm{d}x &= \int_0^1 \frac{x^8-4x^7+6x^6-4x^5+x^4}{1+x^2}\mathrm{d}x \\
&= \int_0^1 x^6-4x^5+5x^4-4x^2+4-\frac{4}{1+x^2} \\
&= \left[\frac{x^7}{7}-\frac{2}{3}x^6+x^5-\frac{4}{3}x^3+4x-4\tan^{-1}(x)\right]_0^1 \\
&= \frac{1}{7}-\frac{2}{3}+1-\frac{4}{3}+4-\pi \\
&= \frac{22}{7}-\pi
\end{aligned}
$$

Notice that this means that $3+\frac{1}{7}=\frac{22}{7}>\pi$, as the integrand is non-negative and so the integral must be positive.

(3) First, notice that the given area is twice the area over $[0,\pi]$ by symmetry. Furthermore, it is clear that the area bounded by the the half-curve and the x-axis plus the area bounded by the half-curve and the y-axis is the area of the rectangle $[0,\pi]\times[0,2]$, which is 2π. Now if the circle rolls with speed 1, the center moves with parametrization $(t,1)$ $(t\in[0,\pi])$. Meanwhile, if we fix our coordinates at the center of the circle, the given point moves with the parametrization $(-\sin t,-\cos t)$ around the center. Thus the path of the point is given by $(t-\sin t, 1-\cos t)$. Therefore, the points on the curve satisfy

$$x = t-\sin t = \cos^{-1}(1-y)-\sqrt{1-(y-1)^2}$$

and so the area bounded by the half-curve and the y-axis is

$$\int_0^2 \cos^{-1}(1-y) - \sqrt{1-(y-1)^2}\mathrm{d}y = \left[(y-1)\cos^{-1}(1-y) + \sqrt{1-(y-1)^2} - \frac{1}{2}\sqrt{1-(y-1)^2} - \frac{1}{2}\sin^{-1}(y-1)\right]_0^2$$
$$= \frac{\pi}{2}$$

and therefore the total area under the curve is

$$2\left(2\pi - \frac{\pi}{2}\right) = 3\pi$$

Alternate route: Parameterize $x(t) = t - \sin t$ and $y(t) = 1 - \cos t$. Since

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 1 - \cos t,$$

$$\int_0^{2\pi} y dx = \int_0^{2\pi} (1-\cos t)\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t == \int_0^{2\pi} (1-\cos t)^2 \mathrm{d}t = 3\pi.$$

Counting Conundrum

(4) $\binom{52}{7} - \left[\binom{13}{6}\binom{39}{1} + \binom{13}{7}\binom{39}{0}\right] = 133,715,920.$

(5) $\dfrac{\binom{12}{1}\binom{2}{2}\binom{11}{3}\binom{2}{1}^3}{\binom{24}{5}} = \dfrac{60}{161}$

(6) Clearly it can't be 0, since there must be 2 people who have the same number of alliances (if someone is allied with no one, there are 4 others and no one can be allied with 4, and similarly if someone is allied with everyone). Here's a list of alliances where only 2 people have the same number of alliances: A is allied with everyone, B is also allied with C, and C and D are also

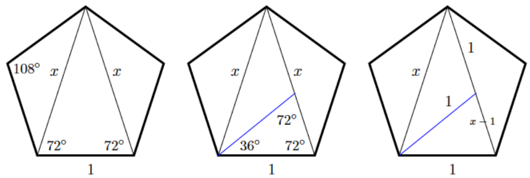
allied. The numbers of alliances are as follows: A has 4, B has 2, C has 3, D has 2, and E has 1. Here the only pair of people who have the same number of alliances is (B,D).

Mystery Mayhem

(7) Christine is missing, either because the fourth speaker recognized Wanda as the first speaker, or either Beth or Julia as the third.

(8) $X = 31,415$. Use the pidgeonhole principle.

(9) The golden ratio, $(1 + \sqrt{5})/2$.



Bisecting the lower angle of 72 degrees, we get another smaller isoceles triangle that is similar to the larger one. Its side lengths are 1, 1 and x-1. Therefore we can set up a proportion

$$\frac{x}{1} = \frac{1}{x-1}$$

$$x^2 - x - 1 = 0$$

$$x = \frac{1 \pm \sqrt{5}}{2}$$

We take the positive solution.