A Whirlwind of Mathematics

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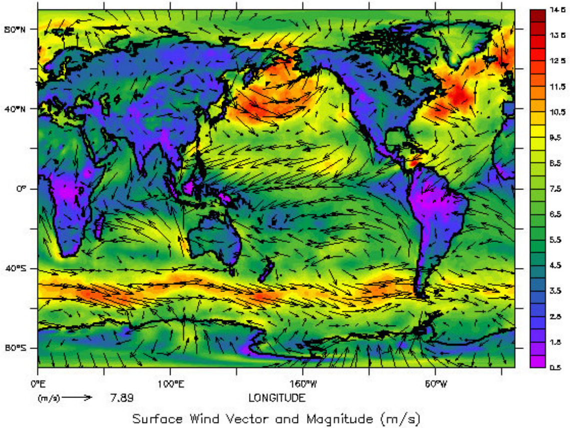
Introduction

A fluid flow can be described by its velocity field $\vec{v} = (u, v, w)$, a 3-dimensional vector field which describes the velocity of the fluid, at each point in space and instant in time.



The Incompressible Navier-Stokes Equations

$$\begin{aligned} \nabla \cdot \vec{v} &= 0 \\ \frac{D\vec{v}}{Dt} &= -\nabla p + \nu \Delta \vec{v} + \vec{F} \end{aligned}$$

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- In general, the sign of $\nabla \cdot \vec{v} < 0$ at a point determines whether the point is (on average) sucking in particles, or pushing them away
- $\nabla \cdot \vec{v} = 0$ is called the incompressibility condition; it enforces the conservation of mass

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- We define the material derivative of a vector-valued function coordinatewise; in particular, $\frac{D\vec{v}}{Dt} = \left(\frac{Du}{Dt}, \frac{Dv}{Dt}, \frac{Dw}{Dt} \right)$

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- The pressure p is a fourth variable to be solved for in the Navier-Stokes equations
- The gradient ∇p is the direction of fastest increase in pressure, and large magnitudes of ∇p correspond to sharper increases in pressure
- Therefore, the equation $\frac{Dv}{Dt} = -\nabla p + [\text{stuff}]$ indicates that fluid particles accelerate away from regions of high pressure

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- The Laplacian applies to $\vec{v}$ coordinatewise: $\Delta\vec{v} = (\Delta u, \Delta v, \Delta w)$

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- We will, as will most textbooks, set $F \equiv 0$

Zoom in on the momentum equation:

The equality in

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$$\begin{aligned}\frac{Du}{Dt} &= -\partial_x p + \nu \Delta u \\ \frac{Dv}{Dt} &= -\partial_y p + \nu \Delta v \\ \frac{Dw}{Dt} &= -\partial_z p + \nu \Delta w\end{aligned}$$

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$$\frac{Dv}{Dt} = -\partial_y p + \nu\Delta v$$
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If we assume that all the action is happening on a 2-dimensional plane (i.e. $w \equiv 0$ and $\partial_z[\text{anything}] = 0$), then the third equation vanishes identically, and we are left with only two momentum equations.

2-dimensional simulation:

2-dimensional WebGL Navier-Stokes simulation

Vorticity:

Define the vorticity $\vec{\omega} = \nabla \times \vec{v}$. Taking the curl of both sides of

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$$\frac{D\vec{\omega}}{Dt} \; = \; (\vec{\omega} \cdot \nabla)\vec{v} + \nu \Delta \vec{\omega}$$

The first term on the right is the vortex-stretching term.

Vorticity in 2 dimensions:

In 2 dimensions, $\vec{\omega} = (0, 0, \partial_x v - \partial_y u)$. Thus, the first two coordinates of the vorticity equation drop out.

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The (comparative) simplicity of this equation indicates that we can expect even more complicated behavior in 3-dimensions.

3-dimensional examples:

Burgers' vortex

3blue1brown's turbulent vortices

Vortex tubes:

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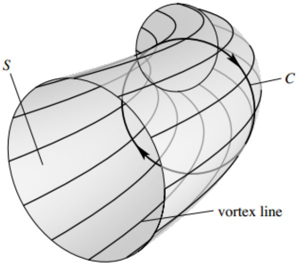
If C is a closed curve, then define the circulation

$$\Gamma_C = \oint_C \vec{v} \cdot d\vec{r}.$$



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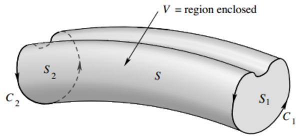

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Let C_1 and C_2 be oriented as in the figure.

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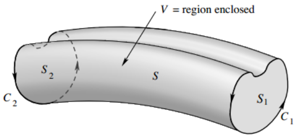

Proof.

Let C_1 and C_2 be oriented as in the figure. The outward flux of vorticity through the enclosed region V is

$$\iint_{S\cup S_1\cup S_2} \vec{\omega}\cdot\vec{n}dA = \iint_{S} \vec{\omega}\cdot\vec{n}dA + \iint_{S_1} \vec{\omega}\cdot\vec{n}dA + \iint_{S_2} \vec{\omega}\cdot\vec{n}dA$$

where $\vec{n}$ is the outward-pointing unit normal to $S \cup S_1 \cup S_2$.

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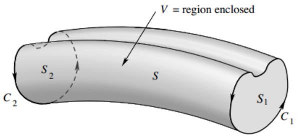

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where $\vec{n}$ is the outward-pointing unit normal to $S \cup S_1 \cup S_2$. The circulation obeys the right-handed rule w.r.t. the normal of S_1, and the left-handed rule w.r.t the normal of S_2.

Helmholtz' theorem (ctd.):

Proof.

By the divergence theorem,

$$\iint_{S \cup S_1 \cup S_2} \vec{\omega} \cdot \vec{n} dA = \iiint_V \nabla \cdot \vec{\omega} d\sigma = \iiint_V \nabla \cdot (\nabla \times \vec{v}) d\sigma = 0.$$

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Proof.

Since the boundary C_1 of a surface S_1 is a closed curve and the circulation obeys the right-handed rule with respect to the normal, Stokes' theorem states that

$$\Gamma_{C_1} = \oint_{C_1} \vec{v} \cdot d\vec{r} = \iint_{S_1} (\nabla \times \vec{v}) \cdot \vec{n} dA = \iint_{S_1} \vec{\omega} \cdot \vec{n} dA.$$

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Similarly, since the circulation about C_2 obeys the left-handed rule,

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Putting this all together,

$$0 = \iint_{S \cup S_1 \cup S_2} \vec{\omega} \cdot \vec{n} dA = \Gamma_{C_1} - \Gamma_{C_2}.$$

$\square$

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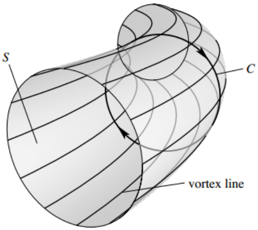
Proof.

Else let the vortex tube end at the boundary of the surface S. On ∂S, $\vec{\omega} \equiv 0$; therefore $\vec{v}$ is conservative on ∂S, and

$$0 = \oint_{\partial S} \vec{v} \cdot \vec{n} = \Gamma_{\partial S}$$



□

One more theorem:

Theorem

In flows with $\nu = 0$, vortex tubes, carried around by the fluid, remain vortex tubes for all time, and maintain their strength.

Parting words:

Whoa
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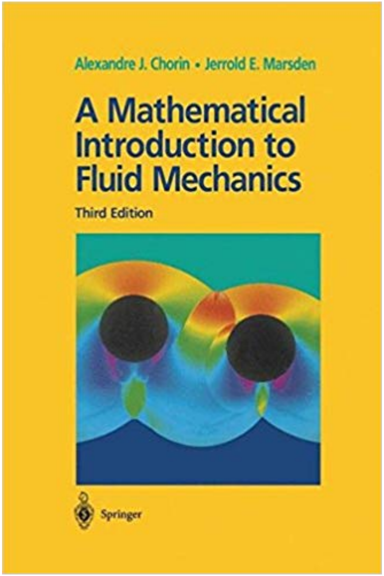
Reference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